HYDROMAGNETIC PULSATIONS OF AN INFINITE CYLINDER-I

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(*Received July* 17, 1968)

ABSTRACT. Radial pulsations of an infinite fluid cylinder have been considered to include the combined effect of finite conductivity and variable density. It is found that, unlike the case of uniform density the period is affected by finite conductivity.

INTRODUCTION

Radial pulsations of an infinite cylinder, for various types of prevailing magnetic fields, have been considered by many authors, (Bhatnagar and Nagpaul, 1957; Chakravorty and Ramamoorthy, 1960, Chandrasekhar and Fermi, 1953, Chopra and Talwar, 1955; Lyttkons, 1954; Raju and Talwar, 1961). The effect of finite conductivity on these radial hydromagnetic pulsations was studied by Bhatnagar and Nagpaul (1957) for a uniform density distribution of the fluid. They showed that finite conductivity does not affect the period of pulsations. It may, therefore, be of some interest to study the combined effect of variable density and finite conductivity on these pulsations. The problem is studied here.

The conductivity σ is assumed to be finite but so large that square and higher powers of $1/\sigma$ are neglected. The density is assumed to decrease from the axis to the surface of the cylinder. The pulsations are assumed to be adiabatic. Under these assumptions this paper includes the discussion for modes higher than the fundamental. A subsequent paper discusses this for the fundamental mode, by a different approach. The study carried out here may be of limited significance because we neglect powers of $1/\sigma$ higher than first and assume an arbitrary variation of density. However, this gives tendencies of the behaviour of the effect of finite conductivity and variable density on the period of pulsations. The order of finite conductivity assumed here is the same as by Bhatnagar and Nagpaul (1957).

EQUATIONS OF THE PROBLEM

The equations governing the small radial adiabatic pulsations of an infinite cylinder have been obtained by Bhatnagar and Nagpaul (1957)
They are :

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{4Gm}{r^2}\,\delta r-\frac{\partial^2}{\partial t^2}\,\delta r\right)+\frac{\partial}{\partial r}\left\{\left(\gamma p+\frac{\mu H^2}{4\pi}\right)\cdot\frac{1}{r}\,\frac{\partial}{\partial r}\,(r\delta r)\right\}\right]$$

$$=\frac{c}{4\pi\mu\sigma}\,\frac{\partial}{\partial r}\left[\frac{1}{r}\,\frac{\partial}{\partial r}\left\{\rho r\left[\frac{4Gm}{r^2}\,\delta r-\frac{\partial^2}{\partial t^2}\,\delta r\right)+\frac{\gamma}{\rho}\,\frac{\partial}{\partial r}\left(\frac{p}{r}\,\frac{\partial}{\partial r}\,r\delta r\right)\right]\right\}\right],\ldots \quad (1)$$

$$\frac{\partial}{\partial t}\left(\frac{\delta H_z}{H}\right)=-\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial}{\partial t}\,\delta r\right)+\frac{c}{\sigma\mu^2H^2}\,\frac{1}{r}\,\frac{\partial}{\partial r}\left[\rho r\left(\frac{4Gm}{r^2}-\frac{\partial^2}{\partial t^2}\right)\delta r+\right.$$

$$\left.+\gamma r\,\frac{\partial}{\partial r}\left\{\frac{p}{r}\,\frac{\partial}{\partial r}\,(r\delta r)\right\}\right]. \quad \ldots (2)$$

In these equations r is the radius of the cylinder at any instant, ρ is the density. p is the pressure. m is the mass per unit length interior to r, δr is the perturbation in r, δH_z is the perturbation in magnetic field $\vec{H}$, γ is the ratio of specific heats assumed to be a constant. μ is the magnetic permeability and G is the gravitational constant.

In deriving these equations the cylinder has been assumed to be immersed in uniform magnetic field parallel to the axis of the cylinder. This is because of the fact that a cylinder cannot be in a steady state in the presence of finite conductivity without volume currents being produced unless the magnetic field in the steady state $\vec{H}$ is uniform. Consequently $\vec{H}$ is taken uniform both inside and outside the cylinder and parallel to the axis of the cylinder.

Equations 1 and 2 hold for all density distributions

CASE OF VARIABLE DENSITY

We take the density to be variable given by :

$$\rho=a-bx^2,\quad x=r/R, \quad (3)$$

where a, b are constants and R is radius of the cylinder. Then

$$m=\pi R^2x^2\left(a-\frac{b}{2}\,x^2\right) \quad (4)$$

For the density distribution of the form (3), the integration of equation of equilibrium

$$\frac{dp}{dx} = -\frac{2Gm}{x}\rho, \qquad .. \quad (5)$$

yields p as

$$p = \pi R^2 G\left[-\frac{b^2}{6}x^6 + \frac{3}{4}abx^4 - a^2x^2 + a^2 - \frac{3}{4}ab + \frac{b^2}{6}\right]$$

$$= \pi R^2 G[d_3x^6 + d_2x^4 + d_1x^2 + d_0]$$

$$= \pi R^2 GD. \qquad .. \quad (6)$$

For the perturbations of the form

$$\psi = \psi_0 \exp(i\omega t), \quad \psi = \frac{\delta r}{R} \qquad .. \quad (7)$$

where ω is the frequency of oscillations, equations (1) and (2) become, using (4)-(6)

$$\frac{1}{\gamma}\left\{4\rho\left(a - \frac{b}{2}x^2\right) + \frac{\omega^2\rho}{\pi G}\right\}\psi_0 + \frac{d}{dx}\left[(D+f)\cdot\frac{1}{x}\frac{d}{dx}(x\psi_0)\right]$$

$$+\frac{iBc}{\sigma}\frac{d}{dx}\left[\frac{1}{x}\frac{d}{dx}\left\{\left[4\rho\left(a - \frac{b}{2}x^2\right) + \frac{\omega^2\rho}{\pi G}\right]\frac{x\psi_0}{\gamma} + \right.\right.$$

$$\left.\left.+x\frac{d}{dx}\left[\frac{D}{x}\frac{d}{dx}(x\psi_0)\right\}\right]\right] = 0, \qquad .. \quad (8)$$

$$\frac{\delta H_z}{H} = -\frac{1}{x}\frac{d}{dx}(x\psi_0) + \frac{Bc}{\sigma fi}\frac{1}{x}\frac{d}{dx}\left[\left\{4\rho\left(a - \frac{b}{2}x^2\right) + \frac{\omega^2\rho}{\pi G}\right\}\frac{x\psi_0}{\gamma} + \right.$$

$$\left.+x\frac{d}{dx}\left\{\frac{D}{x}\frac{d}{dx}(x\psi_0)\right\}\right] \qquad (9)$$

where

$$f \equiv \frac{\mu H^2}{4\pi^2 R^2 G\gamma}, \; B = \frac{1}{4\pi\mu R^2\omega} \qquad .. \quad (10)$$

Equation (8) is to be solved subject to the two boundary conditions :

$$\text{(i)} \quad (\psi_0)_{x=0} = 0, \qquad .. \quad (11)$$

$$\text{(ii)} \quad (\delta P)_{x=1} = 0 \quad \text{i.e.} \quad (\delta H_z)_{x=1} = 0. \qquad .. \quad (12)$$

Writing $\psi_0 = \frac{d\phi}{dx}$ in equation (8) we get, after integration,

$$\frac{1}{\gamma}\int\left\{4\rho\left(a-\frac{b}{2}x^2\right)+\frac{\omega^2\rho}{\pi G}\right\}\frac{d\phi}{dx}dx+\frac{D+f}{x}\left\{\frac{d}{dx}\left(x\frac{d\phi}{dx}\right)\right\}$$

$$+\frac{iBc}{\sigma}\frac{1}{x}\frac{d}{dx}\left[\left\{4\rho\left(a-\frac{b}{2}x^2\right)+\frac{\omega^2\rho}{\pi G}\right\}\frac{x}{\gamma}\frac{d\phi}{dx}\right.$$

$$\left.+x\frac{d}{dx}\left\{\frac{D}{x}\frac{d}{dx}\left(x\frac{d\phi}{dx}\right)\right\}\right]=0. \tag{13}$$

Equation (9) in terms of ϕ is, with the help of (13),

$$\frac{\delta H_z}{H}=\frac{1}{f}\left[\int\left\{4\rho\left(a-\frac{b}{2}x^2\right)+\frac{\omega^2\rho}{\pi G}\right\}\cdot\frac{1}{\gamma}\frac{d\phi}{dx}dx+\frac{D}{x}\frac{d}{dx}\left(x\frac{d\phi}{dx}\right)\right] \quad .. \tag{14}$$

Consequently boundary condition (12) requires that

$$\int\left\{4\rho\left(a-\frac{b}{2}x^2\right)+\frac{\omega^2\rho}{\pi G}\right\}\frac{1}{\gamma}\cdot\frac{d\phi}{dx}dx=0, \text{ at } x=1 \quad .. \tag{15}$$

The quantity within brackets of the integrand in (15) is positive throughout the interval of integration.

Therefore, for the integral (15) to vanish, the amplitude $\frac{d\phi}{dx}$ must assume the value zero somewhere in the interval of integration. The amplitude $\frac{d\phi}{dx}$ can assume the value zero for all the modes higher than the fundamental mode. Therefore the present analysis can be carried out only for modes higher than the fundamental. Integrating (15) by parts we get

$$\frac{\phi}{\gamma}\left\{4\rho\left(a-\frac{b}{2}x^2\right)+\frac{\omega^2\rho}{\pi G}\right\}-\int\frac{\phi}{\gamma}\frac{d}{dx}\left\{4\rho\left(a-\frac{b}{2}x^2\right)+\frac{\omega^2\rho}{\pi G}\right\}dx=0, \text{ at } x=1. \quad .. \tag{16}$$

This will be satisfied only if $\phi = h$ (a finite quantity $\neq 0$) at $x = 1$. Therefore the condition (12) becomes

$$(\phi)_{x=1}=h. \qquad \text{(Finite, non zero)}$$

We include the effect of finite conductivity by defining

$$B_0=\frac{1}{4\pi R^2\mu\omega_0}, \quad \tau=\frac{B_0c}{\sigma}, \quad .. \tag{18}$$

and taking σ to be finite but so large that powers of $1/\sigma$ higher than one may be neglected and therefore retaining quantities of the order of τ only, we can write

$$B = B_0+\tau B_1, \quad \omega = \omega_0+\tau\omega_1, \ \phi = \phi_0+\tau\phi_1, \ A = A_0+\tau A_1, \quad .. \quad (19)$$

where

$$A = \frac{\omega^2}{\pi G\gamma}, \ A_0 = \frac{\omega_0^2}{\pi G\gamma}, \quad A_1 = \frac{2\omega_0\omega_1}{\pi G\gamma}, \quad B_1 = -\frac{\omega_1}{\omega_0} B_0 \quad .. \quad (20)$$

Substituting (19) in (8) and separating the zero and first order terms, we have

$$\left.\begin{aligned} &\left[\frac{4\rho}{\gamma}\left(a-\frac{b}{2}x^2\right)+A_0\rho\right]\frac{d\phi_0}{dx}+\frac{d}{dx}\left[\frac{D+f}{x}\frac{d}{dx}\left(x\frac{d\phi_0}{dx}\right)\right]=0 \\ &\text{with} \qquad \left(\frac{d\phi_0}{dx}\right)_{x=0}=0, \quad (\phi_0)_{x=1}=h_0 \quad \text{(finite)} \end{aligned}\right\} \quad .. \quad (21)$$

and

$$\left.\begin{aligned} &\left[\frac{4\rho}{\gamma}\left(a-\frac{b}{2}x^2\right)+A_0\rho\right]\frac{d\phi_1}{dx}+\frac{d}{dx}\left[\frac{D+f}{x}\frac{d}{dx}\left(x\frac{d\phi_1}{dx}\right)\right] \\ &+A_1\rho\frac{d\phi_0}{dx}-if\frac{d}{dx}\left[\frac{1}{x}\frac{d}{dx}\left\{x\frac{d}{dx}\left[\frac{1}{x}\frac{d}{dx}\left(x\frac{d\phi_0}{dx}\right)\right]\right\}\right]=0, \\ &\text{with} \qquad \left(\frac{d\phi_1}{dx}\right)_{x=0}=0, \quad (\phi_1)_{x=1}=h_1 \ \text{(finite)} \end{aligned}\right\} \quad .. \quad (22)$$

In view of i being present in (22) we conclude that both A_1 and ϕ_1 are complex, Therefore writing

$$\phi_1 = \xi+i\eta, \qquad A_1 = \alpha+i\beta, \quad .. \quad (23)$$

in (22) and taking only the real part in order to study the effect on the period of pulsations (see eqn. (7)) we get

$$\left[\frac{4\rho}{\gamma}\left(a-\frac{b}{2}x^2\right)+A_0\rho\right]\frac{d\xi}{dx}+\alpha\rho\frac{d\phi_0}{dx}+\frac{d}{dx}\left[\frac{D+f}{x}\cdot\frac{d}{dx}\left(x\frac{d\xi}{dx}\right)\right]=0. \quad .. \quad (24)$$

From equation (24) we have to determine α, to study ω_1, by solving (24) for ξ subject to the two boundary conditions :

$$\left(\frac{d\xi}{dx}\right)_{x=0}=0, \quad (\xi)_{x=1}=l \ \text{(finite)}. \quad .. \quad (25)$$

Equation (24) can be rewritten in the form

$$(D+f)\frac{d^3\xi}{dx^3}+\left[\frac{D+f}{x}+\frac{dD}{dx}\right]\frac{d^2\xi}{dx^2}+$$

$$+\left[\frac{1}{x}\frac{dD}{dx}-\frac{D+f}{x^2}+\frac{4\rho}{\gamma}\left(a-\frac{b}{2}x^2\right)+A_0\rho\right]\frac{d\xi}{dx}+\alpha\rho\frac{d\phi_0}{dx}=0 \qquad .. \quad (26)$$

CASE OF INFINITE CONDUCTIVITY

The equation determining ϕ_0, the value of ϕ for the case of infinite conductivity, can be obtained by taking σ to be infinite in equation (8). The reduced eqn. is

$$(D+f)\frac{d^3\phi_0}{dx^3}+\left[\frac{D+f}{x}+\frac{dD}{dx}\right]\frac{d^2\phi_0}{dx^2}+$$

$$+\left[\frac{1}{x}\frac{dD}{dx}-\frac{D+f}{x^2}+4\rho\left(a-\frac{b}{2}x^2\right)+A_0\rho\right]\frac{d\phi_0}{dx}=0. \qquad .. \quad (27)$$

Substituting the values of ρ and D, we attempt a series solution of (27) in the form

$$\phi_0=\sum_{n=0}^{\infty} b_n x^{n+k}, \qquad .. \quad (28)$$

subject to the two boundary conditions :

$$\text{(i)}\ \left(\frac{d\phi_0}{dx}\right)=0, \quad \text{(ii)}\ (\phi_0)_{x=1}=h_0\ \text{(finite)} \qquad .. \quad (29)$$

The indicial equation gives three values for k. The different coefficients b_n satisfy the following recurrence relation between them.

$$Lb_{n-3}+Mb_{n-1}+Nb_{n+1}+Qb_{n+3}=0, \qquad (30)$$

where

$$L=(k+n+1)\left[\{(k+n+2)(k+n+10)+5\}d_3+2\frac{b^2}{\gamma}\right] \qquad (31)$$

$$M=(k+n+1)\left[\{(k+n+2)(k+n+8)+3\}d_2-\frac{6ab}{\gamma}-A_0b\right], \qquad (32)$$

$$N=(k+n+1)\left[\{(k+n+2)(k+n+6)+1\}d_1+\frac{4a^2}{\gamma}+A_0a\right], \qquad (33)$$

$$Q=(d_0+f)[(k+n+1)\{(k+n+2)(k+n+4)-1\}], \qquad (34)$$

Writing

$$\lim_{n \to \infty} \frac{b_{n+1}}{b_{n-1}} = \lambda_0,$$

we have

$$F(\lambda_0) \equiv (d_0+f)\lambda_0^3+d_1\lambda_0^2+d_2\lambda_0+d_3 = 0.$$

Then

$$F(0) = d_3 = -\frac{b^2}{6} = -ve; \qquad F(1) = f = +ve.$$

Therefore $F(\lambda_0) = 0$ has at least one root in the interval $(0, 1)$. Consequently we can say that at least one of the limits λ_0 is less than unity i.e. at least for this limit the series for ϕ_0 would be convergent.

CASE OF FINITE CONDUCTIVITY

For this case equation (26) is to be solved. We seek a series solution of this equation in the form

$$\xi = \sum_{n=0}^{\infty} a_n x^{n+k} \qquad .. \quad (35)$$

where k remains same as in (28) since the indicial equation is same. The recurrence relation in this case is

$$La_{n-3}+Ma_{n-1}+Na_{n+1}+Qa_{n+3}+\alpha(M_1 b_{n-1}+N_1 b_{n+1}) = 0, \qquad .. \quad (36)$$

where L, M, N, Q are same as for the case of infinite conductivity and M_1, N_1 are given by

$$M_1 = -b(k+n-1), \quad N_1 = a(k+n+1).$$

Again if $\lim_{n \to \infty} \frac{a_{n+1}}{a_{n-1}} = \lambda_1$, we see that

$$F(\lambda_1) = (d_0+f)\lambda_1^3+d_1\lambda_1^2+d_2\lambda_1+d_3 = 0$$

This once again shows that at least one of the series for ξ is convergent.

DISCUSSION

In view of the recurrence relation (36), we can write

$$\xi = \frac{a_0}{b_0}\phi_0+\alpha \sum_{n=0}^{\infty} \mu_n x^{n+k}, \qquad . \quad (37)$$

where

$$L\mu_{n-3}+M\mu_{n-1}+N\mu_{n+1}+Q\mu_{n+3}+M_1b_{n-1}+N_1b_{n+1} = 0. \tag{38}$$

In deriving (37) and (38) we have substituted

$$a_n = \alpha\mu_n + \frac{a_0}{b_0}\, b_n. \tag{39}$$

Since at $x = 1$, $\phi_0 = h_0$ and $\xi = l$ [see (25) and (29)] therefore it follows, from (37), that

$$\alpha = \text{finite} \neq 0. \tag{40}$$

Consequently, from (23), we see that both real and imaginary parts of A_1 are non-zero. This implies that ω_1 is complex (See (20)). Therefore the effect of finite conductivity on the frequency of oscillation contains both real and imaginary parts (See. (19)).

We may thus conclude, with the help of (19) and (7), that both amplitude and period of pulsations are affected by finite conductivity in the case of a variable density.

Thus we may say that, unlike the case of uniform density (Bhatnagar and Nagpaul 1957) in which case only the amplitude is exponentially damped, the period is also changed for the case of variable density of the form considered.

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